On the Integration of a System of Differential Equations in Kinematics.

BY JOHN EIESLAND.

If we have a rigid system moving about a fixed point 0 and p, q, r be the components of rotation, in order to determine the motion completely we must integrate the following system of differential equations,

$$(1) \qquad \begin{cases} \dfrac{d\alpha}{dt} = \beta r - \gamma q, \\ \dfrac{d\beta}{dt} = \gamma p - \alpha r, \\ \dfrac{d\gamma}{dt} = \alpha q - \beta p, \end{cases}$$

which has been treated by Euler, D'Alembert, and more recently by Darboux (Theorie des Surfaces, vol. I, Chap. 1–2). Euler expresses the 9 cosines of the variable axes of the system in terms of three angles θ, ψ, ϕ (the so-called Euler's angles) which, if known, will determine the motion of the system.

Darboux reduces the problem of integration to that of a Riccati equation by introducing two new variables x and y which remain constant respectively on each set of rectilinear generators of the sphere $\alpha^2 + \beta^2 + \gamma^2 = 1$ (Leçons, vol. I, p. 22). This method, while very elegant, does not admit of any obvious extension to systems with four and in general n variables such as have been deduced by Professor Craig and Mr. Hatzidakis,* while the introduction of angles corresponding to Euler's angles becomes impracticable for higher dimensions.

In a note published in Am. Jour. vol. 20, I have sketched an integration theory for a system in 4-dimensional space adopting a method employed by Lie to the general linear system in two and three variables. The essential feature

* Am. Jour. of Math., vols. 20 and 23.

of this method is a transformation of the co-ordinates by means of the formulæ

$$\alpha_1 = \frac{\rho_1}{\sqrt{\rho_1^2+\rho_2^2+\ldots\ldots+\rho_{n-1}^2+1}}, \quad \alpha_2 = \frac{\rho_2}{\sqrt{\rho_1^2+\rho_2^2+\ldots\ldots+\rho_{n-1}^2+1}}, \ldots\ldots\ldots,$$

$$\alpha_n = \frac{1}{\sqrt{\rho_1^2+\rho_2^2+\ldots\ldots+\rho_{n-1}^2+1}},$$

which is nothing but an introduction of homogeneous variables. The homogeneous system is thereby transformed into a system of generalized Riccati equations in $n-1$ variables whose coefficients are functions of t as before.

A remark may here be made concerning this transformation. Consider the case of three variables

$$\alpha_1 = \frac{\rho_1}{\sqrt{\rho_1^2+\rho_2^2+1}}, \quad \alpha_2 = \frac{\rho_2}{\sqrt{\rho_1^2+\rho_2^2+1}}, \quad \alpha_3 = \frac{1}{\sqrt{\rho_1^2+\rho_2^2+1}}.$$

If α_1, α_2, α_3 be a point on the unit sphere, then, by central projection of the point on the plane $\alpha_3 = -1$, the corresponding point in this plane will have the co-ordinates ρ_1 and ρ_2, the point of intersection of the α_3 axis with this plane being taken as the origin. To a rotation of the sphere there corresponds a certain motion of the points on the plane, namely a Non-Euclidean motion whose absolute is the circle $\rho_1^2+\rho_2^2+1=0$.*

The transformed system

$$\frac{d\rho_1}{dt} = -q + \rho_2 r + \rho_1(\rho_2 p - \rho_1 q),$$

$$\frac{d\rho_2}{dt} = \quad p - \rho_1 r + \rho_2(\rho_2 p - \rho_1 q),$$

is, however, not easier to integrate than the original system. The only advantage we can draw from this transformation is to render the problem more intuitive, if we intepret ρ_1, ρ_2, t as coordinates of three-dimensional space, (See Lie-Scheffer, Cont. Gr. p. 778), which advantage is lost in the extension of the problem to four or more variables. Moreover, the Euclidean Motion on the sphere is more intuitive for very good reasons than the motion of the Non-Euclidean manifoldness on which the sphere may be projected centrally. We shall therefore not use this transformation, but consider the system in its homogeneous form.

It is the object of this paper to show how by another method that is capable of generalization the integration problem may be reduced to its simplest terms.

* See Klein, Nicht-Euclidische Geometrie, Vorlesungshefte, I, p. 223.

We shall use geometrical language as much as possible and integrate geometrically in the sense employed by Lie in his Berührungstransformationen, and in Lie-Scheffer's volume on continuous groups. The analytical treatment may then be considered as a verification of theorems that have been made evident from geometrical considerations.

I.

We shall begin with the case of two variables and consider the equations

$$
\left\{
\begin{aligned}
\frac{dx_1}{dt} &= \quad p_{12}x_2, \\
\frac{dx_2}{dt} &= -p_{12}x_1,
\end{aligned}
\right.
\tag{2}
$$

where p_{12} is a given function of t. These equations admit of the Euclidean group of rotations around the origin of the plane x_1x_2, that is, the transformation

$$
Uf = -p_{12}\frac{\partial f}{\partial x_1} + p_{12}\frac{\partial f}{\partial x_2};
$$

but since t is a variable we must conceive of the transformation as continually changing with the time t. Now since all the circles $x_1^2 + x_2^2 = \text{const.}$ are invariant curves of the system, the two lines

$$
x_1 = ix_2, \quad x_1 = -ix_2,
$$

which pass through the circular points at infinity are common tangents to all these circles and may therefore be considered as their envelope; hence they are a pair of integral curves of (2). But any system of the form

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha_1 x_1 + \beta_1 x_2, \\
\frac{dx_2}{dt} &= \alpha_2 x_1 + \beta_2 x_2
\end{aligned}
$$

can be solved by one quadrature if two integral curves are known, hence (2) can be solved by one quadrature only.

This may easily be verified analytically. Introducing the variable $\frac{x_1}{x_2} = \omega$ in (2) the system reduces to the form

$$
\frac{d\omega}{dt} = p_{12}(1 + \omega^2)
$$

which is a Riccati equation having $\omega = \pm i$ for particular solutions and can therefore be solved by a single quadrature; the general integral is

$$x_1 = \varkappa \sin \alpha, \quad x_2 = \varkappa \cos \alpha,$$

where

$$\alpha = \int p_{12}\, dt.$$

We shall now consider the system in three variables

$$(3) \qquad \begin{cases} \dfrac{dx_1}{dt} = p_{12}x_2 - p_{13}x_3, \\ \dfrac{dx_2}{dt} = - p_{12}x_1 + p_{23}x_3, \\ \dfrac{dx_3}{dt} = p_{13}x_1 - p_{23}x_2, \end{cases}$$

which is identical with (1), if we put $p_{13} = q$, $p_{12} = r$, $p_{23} = p$. Since the ∞^1 spheres $x_1^2 + x_2^2 + x_3^2 = \text{const.}$ are invariant surfaces, if we determine the motion of any one of these, say the unit sphere, the complete motion is known; we may therefore limit ourselves to the consideration of this sphere.

Let it be supposed that we know an integral curve C on the sphere; we shall prove that the integration of (3) may be reduced to that of the system

$$(4) \qquad \frac{dy_1}{dt} = P_{12}y_2, \quad \frac{dy_2}{dt} = - P_{12}y_1, \quad \frac{dy_3}{dt} = 0.$$

Let the initial position of the x_1, x_2, x_3-axes be such that the x_3-axis passes through a point P_0 on C at the time t_0. We now transform these axes to a set of new but variable axes y_1, y_2, y_3 such that the y_3-axis describes the curve C; this transformation will be a Euclidean one whose 9 direction-cosines are functions of t depending on the given integral curve. We now pass a plane, $y_3 = 0$, through the origin and perpendicular to the y_3-axis and choose arbitrarily in the intersection of this plane with the sphere two points Q and R at a quadrant's distance from each other. As y_3 moves along C, the two axes $OQ = y_1$, $OR = y_2$ will describe a unit circle in the plane $y_3 = 0$. But since the transformation is Euclidean, the transformed system will have the same form in the variables y_1, y_2, y_3 as the original system (3). Moreover, $y_1 = y_2 = 0$ will be a particular solution of the system, that is, the system takes the form (4), *q. e. d.*

To verify this theorem by analysis we proceed as follows:

Let the integral curve be given in the form

$$(5) \qquad x_1 = \rho_1(t)\, x_3, \quad x_2 = \rho_2(t)\, x_3.$$

We transform the axis by means of the orthogonal transformation

$$(6)\qquad \begin{cases} y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\ y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3, \\ y_3 = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3, \end{cases}$$

where the α's are the 9 direction-cosines of the new axis. Now since the y_3-axis always passes through a point P on the integral curve, $y_1 = y_2 = 0$, whenever $x_1 = \rho_1 x_3$ and $x_2 = \rho_2 x_3$; these conditions must then be written

$$\rho_1\alpha_{11} + \rho_2\alpha_{12} + \alpha_{13} \equiv 0,$$
$$\rho_1\alpha_{21} + \rho_2\alpha_{22} + \alpha_{23} \equiv 0,$$

or, more simply,

$$(7)\qquad \rho_1 = \frac{\alpha_{31}}{\alpha_{33}}, \quad \rho_2 = \frac{\alpha_{32}}{\alpha_{33}},$$

that is to say, α_{31}, α_{32}, α_{33} are known functions of t, viz.:

$$(8)\qquad \alpha_{31} = \frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \alpha_{32} = \frac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \alpha_{33} = \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}.$$

Introducing now the new coordinates in (3), we find after easy reductions, keeping always in mind the 6 relations between the 9 cosines α_{ik}

$$\begin{aligned} \frac{dy_1}{dt} = {} & [p_{12}\alpha_{33} - p_{13}\alpha_{32} + p_{23}\alpha_{31} + \alpha_{21}\alpha'_{11} + \alpha_{22}\alpha'_{12} + \alpha_{23}\alpha'_{31}]\, y_2 \\ & + [p_{12}\alpha_{23} - p_{13}\alpha_{22} + p_{23}\alpha_{21} - \alpha'_{31}\alpha_{11} - \alpha'_{32}\alpha_{12} - \alpha'_{33}\alpha_{13}]\, y_3, \\ \frac{dy_2}{dt} = {} & -[p_{12}\alpha_{33} - p_{13}\alpha_{32} + p_{23}\alpha_{31} - \alpha_{21}\alpha'_{11} - \alpha_{22}\alpha'_{12} - \alpha_{23}\alpha'_{13}]\, y_1 \\ & + [p_{12}\alpha_{13} - p_{13}\alpha_{12} + p_{23}\alpha_{11} - \alpha'_{31}\alpha_{21} - \alpha'_{32}\alpha_{22} - \alpha'_{33}\alpha_{23}]\, y_3, \\ \frac{dy_3}{dt} = {} & -[p_{12}\alpha_{23} - p_{13}\alpha_{22} + p_{23}\alpha_{21} - \alpha'_{31}\alpha_{11} - \alpha'_{32}\alpha_{12} - \alpha'_{33}\alpha_{13}]\, y_1 \\ & - [p_{12}\alpha_{13} - p_{13}\alpha_{12} + p_{23}\alpha_{11} - \alpha'_{31}\alpha_{21} - \alpha'_{32}\alpha_{22} - \alpha'_{33}\alpha_{23}]\, y_2, \end{aligned}$$

where the α'_{ik} are derivatives of the α's with respect to t. This system may be written

$$(9)\qquad \begin{cases} \dfrac{dy_1}{dt} = P_{12}y_2 - P_{13}y_3, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3, \\ \dfrac{dy_3}{dt} = P_{13}y_1 - P_{23}y_2, \end{cases}$$

which is of the same form as the original system. But, $P_{13} = P_{23} \equiv 0$; in fact, since by (5) and (7) we have

$$\frac{x_1}{x_3} = \frac{\alpha_{31}}{\alpha_{33}} = \rho_1, \quad \frac{x_2}{x_3} = \frac{\alpha_{32}}{\alpha_{33}} = \rho_2, \tag{10}$$

it follows that $x_1 = \alpha_{31}$, $x_2 = \alpha_{32}$, $x_3 = \alpha_{33}$ are a set of particular solutions of (3) and we must therefore also have

$$\begin{aligned} \alpha'_{31} &= \quad p_{12}\alpha_{32} - p_{13}\alpha_{33}, \\ \alpha'_{32} &= -p_{12}\alpha_{31} + p_{23}\alpha_{33}, \\ \alpha'_{33} &= \quad p_{13}\alpha_{31} - p_{23}\alpha_{32}. \end{aligned}$$

Substituting these values of α'_{31}, α'_{32} and α'_{33} in P_{13} and P_{23}, we find that they vanish identically, and our system reduces to the form

$$\frac{dy_1}{dt} = P_{12}y_2, \quad \frac{dy_2}{dt} = -P_{12}y_1, \quad \frac{dy^3}{dt} = 0, \; q.e.d. \tag{4}$$

If now we determine the six cosines α_{11}, α_{12}, α_{13}; α_{21}, α_{22}, α_{23} in terms of ρ_1 and ρ_2, we know the function P_{12}, so that this last system may be integrated by one quadrature. Having found y_1, y_2 and y_3 we substitute in (6) and solve for x_1, x_2 and x_3 which will be the general integrals of (3). Since the knowledge of an integral curve C on the unit-sphere by (10) amounts to having a set of particular solutions given, we may state the result thus:

If a set of particular solutions of (3) *is known, the system may be integrated by one quadrature.* This is the form in which Darboux states this theorem (Leçons, vol. I, p. 28).

It only remains to show how to calculate the six cosines α_{1i}, α_{2i} $(i = 1, 2, 3)$ in terms of ρ_1 and ρ_2, or, what is the same thing, in terms of α_{31}, α_{32}, α_{33}, which are known functions of t.

The integration problem such as Euler conceived it consists in determining the 9 cosines of the new or variable axes (which are different from the axes y_1, y_2, y_3 chosen above) as functions of three angles ϕ, θ, ψ (see Darboux, vol. I, p. 3). The formulæ thus obtained being rather unsymmetrical, he adopted another system of parameters, introducing the direction-cosines α, β, ν of an axis of rotation a and a rotation ω about the axis. This representation is simplified if we introduce homogeneous parameters putting

$$A = \sin\frac{\omega}{2}\cos\alpha, \quad B = \sin\frac{\omega}{2}\cos\beta, \quad C = \sin\frac{\omega}{2}\cos\gamma, \quad D = \cos\frac{\omega}{2},$$

so that $A^2 + B^2 + C^2 + D^2 = 1$. The nine cosines are then as follows:*

$$\begin{array}{ccc} D^2 + A^2 - B^2 - C^2, & 2(AB - CD), & 2(AC - BD), \\ 2(AB + CD), & D^2 - A^2 + B^2 - C^2, & 2(BC - AD), \\ 2(AC - BD), & 2(BC + AD), & D^2 - A^2 - B^2 + C^2. \end{array}$$

If we put $D = \frac{1}{\Delta}$, $A = \frac{\lambda}{\Delta}$, $B = \frac{\mu}{\Delta}$, $C = \frac{-\nu}{\Delta}$ where $\Delta = 1 + \lambda^2 + \mu^2 + \nu^2$ we obtain the parametric representation of Cayley for $n = 3$.† The table of cosines is then

$$\begin{array}{ccc} \frac{1 + \lambda^2 - \mu^2 - \nu^2}{\Delta}, & 2\frac{\nu + \lambda\mu}{\Delta}, & 2\frac{\lambda\nu - \mu}{\Delta}, \\ 2\frac{-\nu + \lambda\mu}{\Delta}, & \frac{1 + \mu^2 - \nu^2 - \lambda^2}{\Delta}, & 2\left(\frac{\lambda + \mu\nu}{\Delta}\right), \\ 2\frac{\mu + \lambda\nu}{\Delta}, & 2\frac{\nu\mu - \lambda}{\Delta}, & \frac{1 + \nu^2 - \lambda^2 - \mu^2}{\Delta}. \end{array}$$

Now since these cosines are connected by two relations only, viz.:

$$\frac{\alpha_{31}}{\alpha_{33}} = \rho_1, \quad \frac{\alpha_{32}}{\alpha_{33}} = \rho_2,$$

one of these three parameters is perfectly arbitrary and may be put equal to zero; we have then the following table of cosines, putting $\lambda = 0$,

$$\begin{array}{ccc} \frac{1 - \mu^2 - \nu^2}{\Delta}, & \frac{2\nu}{\Delta}, & \frac{-2\mu}{\Delta}, \\ \frac{-2\nu}{\Delta}, & \frac{1 + \mu^2 - \nu^2}{\Delta}, & \frac{2\mu\nu}{\Delta}, \\ \frac{2\mu}{\Delta}, & \frac{2\mu\nu}{\Delta}, & \frac{1 + \nu^2 - \mu^2}{\Delta}, \end{array}$$

where $\Delta = 1 + \mu^2 + \nu^2$. We have now by (7) and (8)

$$\frac{2\mu}{\Delta} = \alpha_{31} = \frac{\rho_1}{\sqrt{1 + \rho_1^2 + \rho_2^2}}, \quad \frac{2\mu\nu}{\Delta} = \alpha_{32} = \frac{\rho_2}{\sqrt{1 + \rho_1^2 + \rho_2^2}}, \quad \frac{1 + \nu^2 - \mu^2}{\Delta} = \alpha_{33} = \frac{1}{\sqrt{1 + \rho_1^2 + \rho_2^2}}.$$

Solving these equations for μ and ν we get

$$\nu = \frac{\rho_2}{\rho_1}, \quad \mu = \frac{1 - \sqrt{1 + \rho_1^2 + \rho_2^2}}{\rho_1}.$$

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* See Encyclopaedie der Math. Wiss., Bd. IV, p. 204.

† Pascal, Die Determinanten p. 162.

Remark. This choice of λ amounts geometrically to choosing the position of the moving axes y_1 and y_2 in the plane $y_3 = 0$ in such a way that the direction-cosines of y_2 become α_{21}, α_{22}, α_{32}, and those of y_1, α_{11}, $-\alpha_{21}$, $-\alpha_{31}$.

Expressing the above cosines in terms of ρ_1 and ρ_2 we obtain the table

$$\begin{array}{ccc}
\dfrac{\rho_1^2 - \rho_2^2\sqrt{1+\rho_1^2+\rho_2^2}}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{\rho_1\rho_2\,(1+\sqrt{1+\rho_1^2+\rho_2^2})}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{-\rho_1}{\sqrt{1+\rho_1^2+\rho_2^2}}, \\
\dfrac{-\rho_1\rho_2\,(1+\sqrt{1+\rho_1^2+\rho_2^2})}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{-\rho_2^2+\rho_1^2\sqrt{1+\rho_1^2+\rho_2^2}}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{+\rho_2}{\sqrt{1+\rho_1^2+\rho_2^2}}, \\
\dfrac{\rho_1}{\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{\rho_2}{\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{1}{\sqrt{1+\rho_1^2+\rho_2^2}},
\end{array}$$

or, in terms of α_{31}, α_{32} and α_{33} which is a set of known particular solutions,

$$\begin{array}{ccc}
\dfrac{\alpha_{33}+\alpha_{31}^2-1}{1-\alpha_{33}}, & \dfrac{\alpha_{31}\alpha_{32}}{1-\alpha_{33}}, & -\alpha_{31}, \\
\dfrac{-\alpha_{31}\alpha_{32}}{1-\alpha_{33}}, & \dfrac{1-\alpha_{32}^2-\alpha_{33}}{1-\alpha_{33}}, & \alpha_{32}, \\
\alpha_{31}, & \alpha_{32}, & \alpha_{33}.
\end{array}$$

Substituting these values in P_{12} of (4) and integrating we find

$$y_1 = x \sin\omega, \quad y_2 = x\cos\omega, \quad y_3 = C.$$

where $\omega = \int P_{12}dt$. Substituting the values of y_1, y_2, y_3 in (6) and solving for the x's we obtain the general integral

$$(11) \qquad \begin{cases} x_1 = \dfrac{\alpha_{33}+\alpha_{31}^2-1}{1-\alpha_{33}}\, x\sin\omega - \dfrac{\alpha_{31}\alpha_{32}}{1-\alpha_{33}}\, x\cos\omega + C\alpha_{31}, \\ x_2 = \dfrac{\alpha_{31}\alpha_{32}}{1-\alpha_{33}}\, x\sin\omega + \dfrac{1-\alpha_{32}^2-\alpha_{33}}{1-\alpha_{33}}\, x\cos\omega + C\alpha_{32}, \\ x_3 = -\alpha_{31}x\sin\omega + \alpha_{32}x\cos\omega + C\alpha_{32}. \end{cases}$$

Suppose now that two integral curves are known. We shall show that the system may be reduced to the form

$$(12) \qquad \frac{dy_1}{dt} = 0 \quad \frac{dy_2}{dt} = 0, \quad \frac{dy^3}{dt} = 0.$$

Let the integral curves be C_1 and C_2. We start from an initial position, placing the extremity of the y_3-axis at a point P on C_1; on C_2 we take a point Q on the y_2-axis at a quadrant's distance from P. As t varies, P moves along C_1 while Q moves along C_2, these curves being integral curves. The axis y_1 will describe an integral curve C_3 which is uniquely determined by the motion, when-

ever C_1 and C_2 are known. The transformed system must now admit $y_1 = y_2 \equiv 0$ and $y_1 = y_3 \equiv 0$ as particular solutions, that is, it must reduce to the form (12), *q. e. d.*

It follows from this theorem that *the system* (3) *may be integrated without quadratures, whenever two integral curves are known.*

Analytically we proceed as follows:

Let the integral curves be

$$x_1 = \rho_1 x_3, \quad x_2 = \rho_2 x_3, \quad x_1 = \sigma_1 x_3, \quad x_2 = \sigma_2 x_3. \tag{13}$$

The transformation (6) must now satisfy the relations

$$\begin{cases} \rho_1 \alpha_{11} + \rho_2 \alpha_{12} + \alpha_{13} \equiv 0, \\ \rho_1 \alpha_{21} + \rho_2 \alpha_{22} + \alpha_{23} \equiv 0, \\ \sigma_1 \alpha_{11} + \sigma_2 \alpha_{12} + \alpha_{13} \equiv 0, \\ \sigma_1 \alpha_{31} + \sigma_2 \alpha_{32} + \alpha_{33} \equiv 0, \end{cases} \tag{14}$$

Since $y_1 = y_2 \equiv 0$ whenever $x_1 = \rho_1 x_3$ and $x_2 = \rho_2 x_3$ and also $y_1 = y_3 \equiv 0$ for $x_1 = \sigma_1 x_3$ and $x_2 = \sigma_2 x_3$. These four conditions reduce to three; in fact, the distance between P and Q must equal $\sqrt{2}$ for all values of t, the initial position having been chosen as explained above. An easy calculation will show that we must have

$$\rho_1 \sigma_1 + \rho_2 \sigma_2 + 1 \equiv 0. \tag{15}$$

The conditions (8) may be written

$$\rho_1 = \frac{\alpha_{31}}{\alpha_{33}}, \quad \rho_2 = \frac{\alpha_{32}}{\alpha_{33}}; \quad \sigma_1 = \frac{\alpha_{21}}{\alpha_{23}}, \quad \sigma_2 = \frac{\alpha_{22}}{\alpha_{23}},$$

which show that $\alpha_{31}, \alpha_{32}, \alpha_{33}; \alpha_{21}, \alpha_{22}, \alpha_{23}$ are two sets of particular solutions. The remaining three cosines are now uniquely determined, so that the table of cosines will be

$$\begin{array}{lll} \dfrac{\sigma_2 - \rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\,\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, & \dfrac{\rho_1 - \sigma_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\,\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, & \dfrac{\rho_2 \sigma_1 - \rho_1 \sigma_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\,\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, \\ \dfrac{\sigma_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, & \dfrac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, & \dfrac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, \\ \dfrac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, & \dfrac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, & \dfrac{1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \end{array}$$

(That these quantities are direction cosines may be verified by aid of (15)).

Transforming now to new axes y_1, y_2, y_3 and reducing we find that P_{13}, P_{23} and P_{12} all vanish so that the system reduces to the form (12). The general integral is now

$$x_1 = \frac{\sigma_2 - \rho_2}{\sqrt{\rho_1^2+\rho_2^2+1}\sqrt{\sigma_1^2+\sigma_2^2+1}}\,x_1 + \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2+1}}\,x_2 + \frac{\rho_1}{\sqrt{\rho_1^2+\rho_2^2+1}}\,x_3,$$

$$x_2 = \frac{\rho_1 - \sigma_1}{\sqrt{\rho_1^2+\rho_2^2+1}\sqrt{\sigma_1^2+\sigma_2^2+1}}\,x_1 + \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2+1}}\,x_2 + \frac{\rho_2}{\sqrt{\rho_1^2+\rho_2^2+1}}\,x_3,$$

$$x_3 = \frac{\rho_2\sigma_1 - \rho_1\sigma_2}{\sqrt{\rho_1^2+\rho_2^2+1}\sqrt{\sigma_1^2+\sigma_2^2+1}}\,x_1 + \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2+1}}\,x_2 + \frac{1}{\sqrt{\rho_1^2+\rho_2^2+1}}\,x_3.$$

II.

We shall now take up the study of the system

$$(1)\qquad \begin{cases} \dfrac{dx_1}{dt} = p_{12}x_2 - p_{13}x_3 + p_{14}x_4, \\[2mm] \dfrac{dx_2}{dt} = -p_{12}x_1 + p_{23}x_3 + p_{24}x_4, \\[2mm] \dfrac{dx_3}{dt} = p_{13}x_1 - p_{23}x_2 + p_{34}x_4, \\[2mm] \dfrac{dx_4}{dt} = -p_{14}x_1 - p_{24}x_2 - p_{34}x_3, \end{cases}$$

which has been obtained by Professor Craig.* Since the hypersphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{const.}$ remains invariant during the motion we may, as before, limit ourselves to the unit sphere. The problem of integration is then: given the six components of rotation p_{ik} around the six planes of the hexahedron of reference, to determine the motion completely. Let a curve C_1 on the hypersphere be known, and let the initial position of the axis at time t_0 be such that the x_4-axis pierces the sphere at a point P on this curve. Transforming to a new system of variable axes y_1, y_2, y_3, y_4 in such a way as to make P describe the curve C_1, as t varies, the motion of the other three axes will take place in a three-dimensional space obtained by cutting the hypersphere by a space $y_4 = 0$ passing through the origin; (this space will necessarily vary in its position with t. It follows then that after the transformation $y_1 = y_2 = y_3 \equiv 0$ must be a particular

* "On the displacement depending on one, two and three variables in space of 4 dimensions," Am. Jour. of Math., vol. 20.

solution of the transformed system. But since a Euclidean transformation does not alter the form of the system (1) we must have

$$(2)\qquad \begin{cases} \dfrac{dy_1}{dt} = \quad P_{12}y_2 - P_{13}y_3\,, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3\,, \\ \dfrac{dy_3}{dt} = \quad P_{13}y_1 - P_{23}y_2\,, \\ \dfrac{dy_4}{dt} = \quad 0\,. \end{cases}$$

Hence the

THEOREM. *If an integral curve of the system* (1) *is known the integration of the system may be reduced to that of a similar system in three variables.*

If an integral curve C'_2 of (2) is known, it is evident that a corresponding integral curve of (1) is known and may be obtained by transforming C'_2 by the inverse of the transformation which changes the system (1) into (2). Let this transformed curve be C_2 and let us suppose it different from C_1; then two integral curves of (1) are known. But we have proved that if an integral curve of a system in three variables is known, it may be integrated by one quadrature; it follows therefore that *the original system can be integrated by one quadrature, whenever two different integral curves are known.*

Since the knowledge of two integral curves amounts to having given two sets of particular solutions, the last theorem may be stated thus:

If two sets of particular solutions of the system (1) *are known, it may be integrated by one quadrature only.*

Since the integration of (2) by Darboux's method* may be reduced to that of an ordinary Riccati equation, the above theorem may also be stated thus:
If one set of particular solutions is known the integration of (1) *may be reduced to that of a Riccati equation.*

The analytical treatment of the integration problem is similar to that employed in the case of ordinary space.

Let the integral curves be

$$(3)\qquad x_1 = \lambda_1(t)\,x_4\,,\quad x_2 = \lambda_2(t)\,x_4\,,\quad x_3 = \lambda_3(t)\,x_4\,,$$

where the λ's are known functions of t. We introduce new axes y_1, y_2, y_3, y_4 by

* Darboux, Theorie des Surfaces, vol. I, p. 22.

means of the Euclidean transformation

$$(4)\qquad \begin{cases} y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4\,, \\ y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \alpha_{24}x_4\,, \\ y_3 = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \alpha_{34}x_4\,, \\ y_4 = \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4\,, \end{cases}$$

where the α's are the 16 direction-cosines of the new axis. Now in order that the point P shall during the motion describe the integral curve (3), we must have $y_1 = y_2 = y_3 = 0$ whenever $x_1 = \lambda_1 x_4$, $x_2 = \lambda_2 x_4$, $x_3 = \lambda_3 x_4$, or

$$(5)\qquad \begin{cases} \alpha_{11}\lambda_1 + \alpha_{12}\lambda_2 + \alpha_{13}\lambda_3 + \alpha_{14} \equiv 0\,, \\ \alpha_{21}\lambda_1 + \alpha_{22}\lambda_2 + \alpha_{23}\lambda_3 + \alpha_{24} \equiv 0\,, \\ \alpha_{31}\lambda_1 + \alpha_{32}\lambda_2 + \alpha_{33}\lambda_3 + \alpha_{34} \equiv 0\,, \end{cases}$$

which may be replaced by the following simple ones:

$$(5^1)\qquad \lambda_1 = \frac{\alpha_{41}}{\alpha_{44}}, \quad \lambda_2 = \frac{\alpha_{42}}{\alpha_{44}}, \quad \lambda_3 = \frac{\alpha_{43}}{\alpha_{44}},$$

that is to say, $x_1 = \alpha_{41}$, $x_2 = \alpha_{42}$, $x_3 = \alpha_{43}$, $x_4 = \alpha_{44}$ is a set of particular solutions of the system (1). Introducing the new coordinates into the system taking into account that the α's are a system of 16 direction-cosines of which the four α_{41}, α_{42}, α_{43}, α_{44} are particular solutions, we obtain after easy reductions

$$(6)\qquad \begin{cases} \dfrac{dy_1}{dt} = P_{12}y_2 - P_{13}y_3\,, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3\,, \\ \dfrac{dy_3}{dt} = P_{13}y_1 - P_{23}y_2\,, \\ \dfrac{dy_4}{dt} = 0\,. \end{cases}$$

where P_{12}, P_{13}, P_{23} have the following values:

$$(7)\qquad \begin{cases} P_{12} = (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\, p_{12} + (\alpha_{21}\alpha_{13} - \alpha_{11}\alpha_{23})\, p_{13} + (\alpha_{11}\alpha_{24} - \alpha_{21}\alpha_{14})\, p_{14} \\ \qquad + (\alpha_{12}\alpha_{23} - \alpha_{22}\alpha_{13})\, p_{23} + (\alpha_{12}\alpha_{24} - \alpha_{22}\alpha_{14})\, p_{24} + (\alpha_{24}\alpha_{14} - \alpha_{23}\alpha_{14})\, p_{34} \\ \qquad + \alpha'_{11}\alpha_{21} + \alpha'_{12}\alpha_{22} + \alpha'_{13}\alpha_{23} + \alpha'_{14}\alpha_{24}\,, \\ P_{13} = (\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\, p_{12} + (\alpha_{11}\alpha_{33} - \alpha_{31}\alpha_{13})\, p_{13} + (\alpha_{31}\alpha_{14} - \alpha_{11}\alpha_{34})\, p_{14} \\ \qquad + (\alpha_{32}\alpha_{13} - \alpha_{12}\alpha_{33})\, p_{23} + (\alpha_{14}\alpha_{32} - \alpha_{12}\alpha_{34})\, p_{24} + (\alpha_{33}\alpha_{14} - \alpha_{13}\alpha_{34})\, p_{34} \\ \qquad + \alpha'_{31}\alpha_{11} + \alpha'_{33}\alpha_{13} + \alpha'_{34}\alpha_{14} + \alpha'_{32}\alpha_{12}\,, \\ P_{23} = (\alpha_{21}\alpha_{32} - \alpha_{31}\alpha_{22})\, p_{12} + (\alpha_{31}\alpha_{23} - \alpha_{33}\alpha_{21})\, p_{13} + (\alpha_{34}\alpha_{21} - \alpha_{31}\alpha_{24})\, p_{14} \\ \qquad + (\alpha_{33}\alpha_{22} - \alpha_{32}\alpha_{23})\, p_{23} + (\alpha_{34}\alpha_{22} - \alpha_{33}\alpha_{24})\, p_{24} + (\alpha_{34}\alpha_{23} - \alpha_{33}\alpha_{24})\, p_{34} \\ \qquad + \alpha_{31}\alpha'_{21} + \alpha_{32}\alpha'_{22} + \alpha_{33}\alpha'_{23} + \alpha_{34}\alpha'_{24}\,. \end{cases}$$

That the coefficients P_{14}, P_{24}, P_{34} vanish identically follows from the fact that α_{41}, α_{42}, α_{43}, α_{44} is a set of particular solutions, so that we have

$$\begin{aligned}
\alpha'_{41} &= p_{12}\alpha_{42} - p_{13}\alpha_{43} + p_{14}\alpha_{44}, \\
\alpha'_{42} &= -p_{12}\alpha_{41} + p_{23}\alpha_{43} + p_{24}\alpha_{44}, \\
\alpha'_{43} &= p_{13}\alpha_{41} - p_{23}\alpha_{42} + p_{34}\alpha_{44}, \\
\alpha'_{44} &= -p_{14}\alpha_{41} - p_{24}\alpha_{42} - p_{34}\alpha_{43};
\end{aligned}$$

Substituting these values of α'_{41}, α'_{42}, α'_{43}, α'_{44} in P_{14}, P_{24} and P_{34} they vanish identically.

Since the α's may be expressed as functions of 6 parameters and since by (5) or (5^1) we have determined 4 of the 16 cosines as functions of λ_1, λ_2 and λ_3, there remain 3 arbitrary parameters which may be chosen entirely at pleasure. In the case of a system in three variables we showed how to actually calculate the simplest system of cosines in terms of the known functions ρ_1 and ρ_2. The corresponding problem for a system in four variables offers no difficulties. If we express the 16 cosines α_{ik} in terms of 6 parameters s_1, s_2, s_3, s_4, s_5, s_6 and then put $s_4 = s_5 = s_6 = 0$ we obtain the following system of cosines:*

$$(8)\quad \left\{\begin{array}{llll}
\dfrac{1 - s_1^2 - s_2^2 - s_3^2}{\Delta}, & \dfrac{2s_1}{\Delta}, & \dfrac{-2s_2}{\Delta}, & \dfrac{2s_3}{\Delta}, \\[2ex]
\dfrac{-2s_1}{\Delta}, & \dfrac{1 + s_2^2 + s_3^2 - s_1^2}{\Delta}, & \dfrac{2s_1s_2}{\Delta}, & \dfrac{-2s_1s_3}{\Delta}, \\[2ex]
\dfrac{2s_2}{\Delta}, & \dfrac{2s_1s_2}{\Delta}, & \dfrac{1 + s_1^2 + s_3^2 - s_2^2}{\Delta}, & \dfrac{2s_2s_3}{\Delta}, \\[2ex]
\dfrac{-2s_3}{\Delta}, & \dfrac{-2s_1s_3}{\Delta}, & \dfrac{2s_2s_3}{\Delta}, & \dfrac{1 + s_1^2 + s_2^2 - s_3^2}{\Delta},
\end{array}\right.$$

where $\Delta = 1 + s_1^2 + s_2^2 + s_3^2$. Now since α_{41}, α_{42}, α_{43}, α_{44} is a set of particular solutions we have

$$\frac{2s_3}{\Delta} = \alpha_{41}, \quad \frac{-2s_1s_3}{\Delta} = \alpha_{42}, \quad \frac{2s_2s_3}{\Delta} = \alpha_{43}, \quad \frac{1 + s_1^2 + s_2^2 - s_3^2}{\Delta} = \alpha_{44};$$

Solving these equations for s_1, s_2 and s_3 we find

$$s_1 = \frac{\alpha_{42}}{\alpha_{41}}, \quad s_2 = \frac{-\alpha_{43}}{\alpha_{41}}, \quad s_3 = \frac{\alpha_{44} - 1}{\alpha_{41}};$$

---

* See Pascal, Die Determinanten, pp. 160–162.

the table (8) may now be written

$$
\begin{array}{cccc}
\dfrac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}, & \dfrac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}, & \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}, & -\alpha_{41}, \\
\dfrac{-\alpha_{42}\alpha_{41}}{1-\alpha_{44}}, & \dfrac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}, & \dfrac{-\alpha_{42}\alpha_{43}}{1-\alpha_{44}}, & \alpha_{42}, \\
\dfrac{-\alpha_{43}\alpha_{41}}{1-\alpha_{44}}, & \dfrac{-\alpha_{42}\alpha_{43}}{1-\alpha_{44}}, & \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}, & \alpha_{43}, \\
\alpha_{41}, & \alpha_{42}, & \alpha_{43}, & \alpha_{44},
\end{array}
$$

and we thus arrive at the following orthogonal transformation:

$$
(9)\quad \begin{cases}
y_1 = \dfrac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}x_1 + \dfrac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}x_2 + \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}x_3 - \alpha_{41}x_4, \\
y_2 = \dfrac{-\alpha_{41}\alpha_{42}}{1-\alpha_{44}}x_1 + \dfrac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}x_2 - \dfrac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}x_3 + \alpha_{42}x_4, \\
y_3 = \dfrac{-\alpha_{43}\alpha_{41}}{1-\alpha_{44}}x_1 - \dfrac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}x_2 + \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}x_3 + \alpha_{43}x_4, \\
y_4 = \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4,
\end{cases}
$$

by means of which the system (1) *is transformed into the system* (6).

Now suppose we know a set of particular solutions of (6), or, to use geometrical language, an integral curve on the sphere $y_1^2+y_2^2+y_3^2=1$, $y_4=0$. Since the first three equations (6) are independent of the last, we consider these equations as defining a rotation of the sphere $y_1^2+y_2^2+y_3^2=\text{const.}$ around a fixed point 0. This case has been treated in the first part of this paper, where we found that such a system is integrable by one quadrature, a set of particular solutions being known.

Now I say that *if a set of particular solutions of* (6) *is known, a set of particular solutions of* (1) *is known and conversely*. The proof is immediate, for, let β_{31}, β_{32}, β_{33} 0, be such a set; substituting these for y_1, y_2, y_3, y_4 respectively in (9) and solving for the x's we obtain

$$
(10)\quad \begin{cases}
x_1^0 = \dfrac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}\beta_{31} - \dfrac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}}\beta_{32} - \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\beta_{33}, \\
x_2^0 = \dfrac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}\beta_{31} + \dfrac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{32} - \dfrac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\beta_{33}, \\
x_3^0 = \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\beta_{31} - \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\beta_{32} + \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{33}, \\
x_4^0 = -\alpha_{41}\beta_{31} + \alpha_{42}\beta_{32} + \alpha_{43}\beta_{33},
\end{cases}
$$

which are easily seen to satisfy the relation $x_1^{0^2} + x_2^{0^2} + x_3^{0^2} + x_4^{0^2} = 1$. Morever, since $y_4 = 0$ is a particular solution of the last of equations (6) we have from the last of equations (9)

$$y_4 = 0 = \alpha_{41}x_1^0 + \alpha_{42}x_2^0 + \alpha_{43}x_3^0 + \alpha_{44}x_4^0,$$

which means that $x_1^0, x_2^0, x_3^0, x_4^0$ are four direction-cosines on the hypersphere.

Conversely, suppose given two sets of particular solutions of (1), viz.:

$$\alpha_{41}\alpha_{42}\alpha_{43}\alpha_{44}\,;\ x_1^0,\ x_2^0,\ x_3^0,\ x_4^0,$$

and if we transform (1) by means of (9) into the system (6), this system will have for particular solutions $\beta_{31}, \beta_{32}, \beta_{33}$, 0 obtained by substituting $x_1^0, x_2^0, x_3^0, x_4^0$ for the x's in (9), that is, we will have

$$y_1 = \beta_{31},\ y_2 = \beta_{32},\ y_3 = \beta_{33},\ y_4 = 0,\ q.e.d.$$

We are now ready to complete the problem. Transforming (6) to a system of new axes z_1, z_2, z_3 by means of the transformation

$$(11)\qquad \begin{cases} z_1 = \dfrac{\beta_{33} + \beta_{31}^2 - 1}{1 - \beta_{33}}\, y_1 + \dfrac{\beta_{31}\beta_{32}}{1 - \beta_{33}}\, y_2 - \beta_{31}y_3, \\ z_2 = \dfrac{-\beta_{31}\beta_{32}}{1 - \beta_{33}}\, y_1 + \dfrac{1 - \beta_{33} - \beta_{32}^2}{1 - \beta_{33}}\, y_2 + \beta_{32}y_3, \\ z_3 = \beta_{31}y_1 + \beta_{32}y_2 + \beta_{33}y_3, \\ z_4 = y_4, \end{cases}$$

we obtain a new system of the form

$$(12)\qquad \frac{dz_1}{dt} = \overline{P}_{12}z_2, \quad \frac{dz_2}{dt} = -\overline{P}_{12}z_1, \quad \frac{dz_3}{dt} = 0, \quad \frac{dz_4}{dt} = 0,$$

which may be integrated by one quadrature. The general integral of (1) may now be easily obtained. The general integral of (6) is, (compare (11) part I)

$$\begin{aligned} y_1 &= \frac{\beta_{33} + \beta_{31}^2 - 1}{1 - \beta_{33}}\, \varkappa \sin \bar{\omega} - \frac{\beta_{31}\beta_{32}}{1 - \beta_{33}}\, \varkappa \cos \bar{\omega} + C\beta_{31}, \\ y_2 &= \frac{\beta_{31}\beta_{32}}{1 - \beta_{33}}\, \varkappa \sin \bar{\omega} + \frac{1 - \beta_{33} - \beta_{32}^2}{1 - \beta_{33}}\, \varkappa \cos \bar{\omega} + C\beta_{32}, \\ y_3 &= -\beta_{31}\, \varkappa \sin \bar{\omega} + \beta_{32}\, \varkappa \cos \bar{\omega} + C\beta_{33}, \\ y_4 &= \text{const.} \end{aligned}$$

where $\bar{\omega} = \int \overline{P}_{12}\,dt$. Substituting these values in (9) and solving for the x's we

obtain the general integral

$$
(13)\left\{
\begin{aligned}
x_1 = {} & \left\{\frac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}\cdot\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}-\frac{\alpha_{41}\alpha_{42}\beta_{31}\beta_{32}}{(1-\alpha_{44})(1-\beta_{33})}\right.\\
& \qquad\left.+\frac{\alpha_{43}\alpha_{41}\beta_{31}}{1-\alpha_{44}}\right\}\varkappa\sin\bar{\omega}\\
& -\left\{\frac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}+\frac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}}\cdot\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\right.\\
& \qquad\left.-\frac{\alpha_{43}\alpha_{41}\beta_{32}}{1-\alpha_{44}}\right\}\varkappa\cos\bar{\omega}\\
& +\ c x_1^0+\text{const. }\alpha_{41}.\\
x_2 = {} & \left\{\frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta}+\frac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\right.\\
& \qquad\left.+\frac{\alpha_{42}\alpha_{43}\beta_{31}}{1-\alpha_{44}}\right\}\varkappa\sin\bar{\omega}\\
& -\left\{\frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}-\frac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}\cdot\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\right.\\
& \qquad\left.+\frac{\alpha_{42}\alpha_{43}\beta_{32}}{1-\alpha_{44}}\right\}\varkappa\cos\bar{\omega}\\
& +\ c x_2^0+\text{const. }\alpha_{42}.\\
x_3 = {} & \left\{\frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}-\frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\right.\\
& \qquad\left.-\frac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{31}\right\}\varkappa\sin\bar{\omega}\\
& -\left\{\frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}+\frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\cdot\frac{1-\beta_{33}\beta_{32}^2}{1-\beta_{33}}\right.\\
& \qquad\left.-\frac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{32}\right\}\varkappa\cos\bar{\omega}\\
& +\ c x_3^0+\text{const. }\alpha_{43}.\\
x_4 = {} & -\left\{\alpha_{41}\cdot\frac{\beta_{33}+\beta_{32}^2-1}{1-\beta_{33}}-\frac{\alpha_{42}\beta_{31}\beta_{32}}{1-\beta_{33}}+\alpha_{43}\beta_{31}\right\}\varkappa\sin\bar{\omega}\\
& +\left\{\frac{\alpha_{41}\beta_{31}\beta_{32}}{1-\beta_{33}}-\frac{\alpha_{42}\beta_{31}\beta_{32}}{1-\beta_{33}}+\alpha_{43}\beta_{32}\right\}\varkappa\cos\bar{\omega}\\
& +\ c x_4^0+\text{const. }\alpha_{44}.
\end{aligned}
\right.
$$

In these formulæ the values of β_{31}, β_{32}, β_{33} are to be replaced by their values in terms of x_1^0, x_2^0, x_3^0, x_4^0 obtained from equations (10). The determination of the angle ω is the only quadrature involved, otherwise the work has been purely algebraic.*

*It should be noticed that the system of integrals (13) contain 4 constants of integration: κ, C, const. and a fourth is obtained in the quadrature $\int \bar{P}_{12}\,dt$.

It is now comparatively easy to obtain an orthogonal transformation which shall transform (1) directly into (12). In fact, combining (9) and (11) we obtain after somewhat lengthy reductions:

$$
(14)\quad \left\{
\begin{aligned}
z_1 = {} & \left\{ \frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}} \cdot \frac{\alpha_{44}+\alpha_{41}^2-1}{1-\alpha_{44}} - \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \frac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}} + \frac{\beta_{31}\alpha_{43}\alpha_{41}}{1-\alpha_{44}} \right\} x_1 \\
& + \left\{ \frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}} \cdot \frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}} + \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \frac{1-\alpha_{44}^2-\alpha_{44}}{1-\alpha_{44}} + \frac{\beta_{31}\alpha_{42}\alpha_{43}}{1-\alpha_{44}} \right\} x_2 \\
& + \left\{ \frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}} \cdot \frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}} - \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}} - \frac{\beta_{31}(1-\alpha_{43}^2-\alpha_{44})}{1-\alpha_{44}} \right\} x_3 \\
& - \left\{ \frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}} \cdot \alpha_{41} - \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \alpha_{42} + \beta_{31}\alpha_{43} \right\} x_4 . \\
z_2 = {} & - \left\{ \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \frac{\alpha_{44}^2+\alpha_{41}^2-1}{1-\alpha_{44}} + \frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}} \cdot \frac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}} + \frac{\beta_{32}\alpha_{43}\alpha_{41}}{1-\alpha_{44}} \right\} x_1 \\
& - \left\{ \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}} - \frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}} \cdot \frac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}} + \frac{\beta_{32}\alpha_{43}\alpha_{42}}{1-\alpha_{44}} \right\} x_2 \\
& - \left\{ \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}} + \frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}} \cdot \frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}} - \frac{\beta_{32}(1-\alpha_{43}^2-\alpha_{44})}{1-\alpha_{44}} \right\} x_3 \\
& + \left\{ \frac{\beta_{31}\beta_{32}}{1-\beta_{33}} \cdot \alpha_{41} + \frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}} \cdot \alpha_{42} + \beta_{32}\alpha_{43} \right\} x_4 . \\
z_3 = {} & x_1^0 x_1 + x_2^0 x_2 + x_3^0 x_3 + x_4^0 x_4 , \\
z_4 = {} & \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4 .
\end{aligned}
\right.
$$

In the first two equations β_{31}, β_{32}, β_{33} have the same values as in (13). This transformation exhibits a system of direction-cosines depending on the 8 parameters α_{41}, α_{42}, α_{43}, α_{44}, x_1^0, x_2^0, x_3^0, x_4^0 between which 3 relations exist. We have thus incidentally solved the problem: *of a set of* 16 *direction-cosines* 8 *are given; it is required to calculate the remaining* 8 *in terms of the given parameters. The solution of this problem is equivalent to reducing the integration of* (1) *to a single quadrature, when two sets of particular solutions are known.* It should be noticed that this problem can be solved in an infinite number of ways, the simplest set of cosines being obtained by putting $s_4 = s_5 = s_6 = 0$. *There exist then* ∞^3 *Euclidean transformations which will effect the reduction of the system* (1) *to the form*

$$
\frac{dz_1}{dt} = P_{12}z_2 , \quad \frac{dz_2}{dt} = - P_{12}z_1 , \quad \frac{dz_3}{dt} = 0 , \quad \frac{dz_3}{dt} = 0 .
$$

The case where three sets of particular solutions are known (or three integral curves) requires no special consideration; a Euclidean transformation may then

be found which will transform (1) into the system

$$\frac{dy_1}{dt}=0\,, \quad \frac{dy_2}{dt}=0\,, \quad \frac{dy_3}{dt}=0\,, \quad \frac{dy_4}{dt}=0\,.$$

which requires no quadrature. If $\alpha_{41}, \alpha_{42}, \alpha_{43}, \alpha_{44}; \alpha_{31}, \alpha_{32}, \alpha_{33}, \alpha_{34}; \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}$ be the given sets, the transformation may be written down at once as follows:

$$\begin{aligned} y_1 &= D_{11}x_1 + D_{12}x_2 + D_{13}x_3 + D_{14}x_4\,, \\ y_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \alpha_{24}x_4\,, \\ y_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \alpha_{34}x_4\,, \\ y_4 &= \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4\,, \end{aligned}$$

where $D_{11}, D_{12}, D_{13}, D_{14}$ are the minors of $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}$ in the determinant of the 16 cosines $|\,\alpha_{11}, \alpha_{22}, \alpha_{33}, \alpha_{44}\,|$.

III.

There remains yet the problem of generalizing the preceding theory of integration. Mr. Hatzidakis has obtained the general system defining the rotation of a rigid system in n-dimensional space around a fixed point.* This system may with slight modifications affecting only the signs of the coefficients p_{ik} be be written †

$$(1) \qquad \begin{cases} \dfrac{dx_1}{dt} = \phantom{-p_{12}x_1+{}} p_{12}x_2 + p_{13}x_3 + \cdots\cdots\cdots + p_{12}x_n\,, \\ \dfrac{dx_2}{dt} = -p_{12}x_1 + p_{23}x_3 + \cdots\cdots\cdots + p_{2n}x_n\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{dx_n}{dt} = -p_{1n}x_1 - p_{2n}x_2 - \cdots\cdots\cdots - p_{n-1n}x_{n-1}. \end{cases}$$

It will be easy to prove as in the case of four dimensions that *the integration of this system may be reduced to that of a similar system, but of one dimension less, whenever a set of particular solutions is known.* The given integral curve being C_1 we transform to a new set of axes $y_1, y_2, \ldots\ldots y_n$ in such a way that the y_n-axis pierces the hypersphere $\Sigma x_n^2 = 1$ at a point P on the given integral curve. In

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* Am. Jour. of Math., vol. 23.

† See my article "On Nullsystems in Space of 5 Dimensions and their Relation to Ordinary Space," Am. Jour. of Math., vol. 25, p. 105. The p's in this article are constants, while here they are given functions of t; moreover, n is not restricted to an odd integer; n may be even or odd.

the transformed system $y_1 = y_2 = y_3 = \ldots . y_{n-1} = 0$ must be a particular solution, so that the last equation reduces to the form $\frac{dy_n}{dt} = 0$, and the first $n - 1$ equations will define a rotation of the sphere $y_1^2 + y_2^2 + \ldots . y_{n-1}^2 = C, y_n = 0$ about the fixed point 0. In general, if m integral curves are known, we may transform the system by means of a Euclidean transformation in such a way that y_n describes C_1, y_{n-1} describes $C_2 \ldots .$ and y_{n-m} describes $C_m \ldots .$ The extremities of the remaining axis will move along paths on the $n - m$ dimensional sphere $y_1^2 + y_2^2 + \ldots . y_{n-m}^2 = 1$, $y_{n-m+1} = y_{n-m+2} = \ldots . y_n = 0$, and the system (1) will be reduced to the form

$$
(2) \quad \begin{cases}
\frac{dy_1}{dt} = P_{12}y_2 + P_{13}y_3 + \ldots\ldots\ldots + P_{1n-m}y_{n-m}, \\
\frac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3 + \ldots\ldots\ldots + P_{2n-m}y_{n-m}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{dy_{n-m}}{dt} = -P_{1n-m}y_1 - P_{2n-m}y_2 - \ldots\ldots\ldots - P_{n-m-1,n-m}y_{n-m-1}, \\
\frac{dy_{n-m+1}}{dt} = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{dy_n}{dt} = 0.
\end{cases}
$$

In particular, *if $n - 2$ integral curves are known the system may be reduced to the form*

$$\frac{dy_1}{dt} = P_{12}y_2, \quad \frac{dy_2}{dt} = -P_{12}y_1, \quad \frac{dy_3}{dt} = 0, \quad \ldots\ldots\ldots\ldots, \quad \frac{dy_n}{dt} = 0,$$

and may therefore be integrated by one quadrature.

We shall show how this reduction may be effected by means of a series of Euclidean transformations. The problem consists in determining a set of n^2 cosines in terms of a given set of particular solutions, n being any given integer > 4.

Let the integral curve be

$$x_1 = \rho_1 x_n, \quad x_2 = \rho_2 x_n, \quad \ldots\ldots\ldots, \quad x_{n-1} = \rho_{n-1} x_n.$$

Since on the unit hypersphere $x_1^2 + x_2^2 + \ldots\ldots x_n^2 = 1$, this is equivalent to the knowledge of a set of particular solutions

$$\frac{\rho_1}{\sqrt{1 + \Sigma\rho_{n-1}^2}}, \quad \frac{\rho_2}{\sqrt{1 + \Sigma\rho_{n-1}^2}}, \quad \ldots\ldots\ldots \frac{\rho_{n-1}}{\sqrt{1 + \Sigma\rho_{n-1}^2}}, \quad \frac{1}{\sqrt{1 + \Sigma\rho_{n-1}^2}},$$

We employ the orthogonal transformation

$$(3) \qquad \begin{cases} y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \ldots\ldots\ldots + \alpha_{1n}x_n, \\ y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \ldots\ldots\ldots + \alpha_{2n}x_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \ldots\ldots\ldots + \alpha_{nn}\alpha_n. \end{cases}$$

In the same way as before in the case of $n = 4$, $y_1 = y_2 = \ldots\ldots = y_{n-1}$ must vanish, whenever $x_1 = \rho_1 x_n$, $x_2 = \rho_2 x_n$, $\ldots\ldots x_{n-1} = \rho_{n-1}x_n$, and hence the following conditions must be satisfied:

$$\begin{aligned} &\alpha_{11}\rho_1 + \alpha_{12}\rho_2 + \ldots\ldots\ldots + \alpha_{1n-1}\rho_{n-1} + \alpha_{1n} \equiv 0, \\ &\alpha_{21}\rho_1 + \alpha_{22}\rho_2 + \ldots\ldots\ldots + \alpha_{2n-1}\rho_{n-1} + \alpha_{2n} \equiv 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\alpha_{n-11}\rho_1 + \alpha_{n-12}\rho_2 + \ldots\ldots\ldots + \alpha_{n-1n-1}\rho_{n-1} + \alpha_{n-1n} \equiv 0, \end{aligned}$$

which $n - 1$ conditions may be reduced to the form

$$(4) \qquad \rho_1 = \frac{\alpha_{n1}}{\alpha_{nn}}, \quad \rho_2 = \frac{\alpha_{n2}}{\alpha_{nn}}, \quad \ldots\ldots\ldots \rho_{n-1} = \frac{\alpha_{nn-1}}{\alpha_{nn}},$$

that is, $\alpha_{n1}, \alpha_{n2}, \ldots\ldots \alpha_{nn}$ must be the given set of particular solutions. In order to obtain a transformation that will reduce the system (1), which we shall denote by Λ_n, to a system Λ_{n-1}, we must find a table of cosines α_{ik} expressed in terms of the n cosines α_{ni}, $(i = 1, 2 \ldots\ldots n)$. Since these n cosines are determined as functions of $\rho_1, \rho_2 \ldots\ldots \rho_{n-1}$, there remain $\frac{n.\overline{n-1}}{2} - (n-1)$ parameters which may be disposed of at will. Let the $\frac{n.\overline{n-1}}{2}$ parameters be denoted by $s_1, s_2, \ldots\ldots s_{\frac{n.\overline{n-1}}{2}}$. We now put all except the first $n - 1$ of these equal to zero and calculate the n^2 direction-cosines in terms of these parameters by the

ordinary method. (Pascal, Die Determinanten, pp. 160–161). We find after easy calculations the following table,

$$\frac{1-s_1^2-s_2^2-\ldots s_{n-1}^2}{\Delta},\quad \frac{2s_1}{\Delta},\quad \frac{-2s_2}{\Delta},\quad \ldots\ldots\ldots \frac{(-1)^n 2s_{n-1}}{\Delta},$$

$$\frac{-2s_1}{\Delta},\quad \frac{1+s_2^2+\ldots+s_{n-1}^2-s_1^2}{\Delta},\quad \frac{2s_1s_2}{\Delta},\quad \ldots\ldots\ldots \frac{(-1)^{n-1}2s_1s_{n-1}}{\Delta},$$

$$\frac{2s_2}{\Delta},\quad \frac{2s_1s_2}{\Delta},\quad \frac{1+s_2^2+s_3^2+s_4^2+\ldots+s_{n-1}^2-s_2^2}{\Delta},\quad \ldots\ldots\ldots \frac{(-1)^{n-2}2s_2s_{n-1}}{\Delta},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{(-1)^{n-1}2s_{n-1}}{\Delta},\quad \frac{(-1)^{n-1}2s_1s_{n-1}}{\Delta},\quad \frac{(-1)^{n-2}2s_2s_{n-1}}{\Delta},\quad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots \frac{1+s_1{}^2+s_2{}^2+\ldots+s_{n-1}^2-s_{n-1}^2}{\Delta},$$

where $\Delta = 1 + \sum_1^{n-1} s_n^2$. We have then

$$\frac{(-1)^{n-1}2s_{n-1}}{\Delta} = \alpha_{n1},\quad \frac{(-1)^{n-1}2s_1s_{n-1}}{\Delta} = \alpha_{n2},\quad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots \frac{1+s_1^2+\ldots+s_{n-2}^2-s_{n-1}^2}{\Delta} = \alpha_{nn},$$

solving which for $s_1, s_2, \ldots s_{n-1}$ we obtain

$$s_1 = \frac{\alpha_{n2}}{\alpha_{n1}},\quad s_2 = \frac{-\alpha_{n3}}{\alpha_{n1}},\quad s_3 = +\frac{\alpha_{n4}}{\alpha_{n1}},\quad \ldots\ldots\ldots \quad s_{n-1} = \frac{\alpha_{nn}-1}{\alpha_{n1}},$$

and we have the following transformation:

$$(5)\quad \begin{cases} y_1 = \dfrac{\alpha_{n1}^2+\alpha_{nn}-1}{1-\alpha_{nn}}x_1 + \dfrac{\alpha_{n2}\alpha_{n1}}{1-\alpha_{nn}}x_2 + \ldots + \dfrac{\alpha_{nn-1}\alpha_{n1}}{1+\alpha_{nn}}x_{n-1} + (-1)^n\alpha_{n1}x_n, \\ y_2 = -\dfrac{\alpha_{n2}\alpha_{n1}}{1-\alpha_{nn}}x_1 + \dfrac{1-\alpha_{n2}^2-\alpha_{nn}}{1-\alpha_{nn}}x_2 + \ldots - \dfrac{\alpha_{nn-1}\alpha_{n2}}{1+\alpha_{nn}}x_{n-1} + (-1)^{n-1}\alpha_{n2}x_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \ldots\ldots\ldots\ldots \alpha_{nn-1}x_3 + \alpha_{nn}x_4, \end{cases}$$

which transforms (1) into the form

$$
(6) \quad \left\{
\begin{aligned}
\frac{dy_1}{dt} &= P_{12}y_2 + P_{13}y_3 + \cdots\cdots + P_{1n-1}y_{n-1}, \\
\frac{dy_2}{dt} &= - P_{12}y_1 + P_{23}y_3 + \cdots\cdots + P_{2n-1}y_{n-1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{dy_{n-1}}{dt} &= - P_{1n-1}y_1 - P_{2n-2}y_2 - \cdots\cdots - P_{n-2n-1}y_{n-2}, \\
\frac{dy_n}{dt} &= 0.
\end{aligned}
\right.
$$

By a method analogous to the one we employed in the case of $n = 4$ we may prove that if a set of particular solutions of (6) is known, a set of solutions of (1) is also known and conversely. Treating the system (6) in the same manner as (1) we may reduce it to another system the integration of which is equivalent to that of a system of dimension $n - 2$. The Euclidean transformation transforming Λ_n into a system Λ_{n-2} is found by combining the two transformations transforming Λ_n into Λ_{n-1} and Λ_{n-1} into Λ_{n-2}. In general, calling the successive systems $\Lambda_n, \Lambda_{n-1}, \ldots. \Lambda_{n-m}$, and the corresponding transformations $S_n, S_{n-1}, \ldots S_{n-m+1}$, we may by means of a transformation $S = S_n S_{n-1} \ldots S_{n-m+1}$ transform Λ_n into Λ_{n-m}. In particular, if $m = n - 2$ the transformation $S = S_n S_{n-1} \ldots. S_3$ will transform Λ_n into the system Λ_2 or,

$$
\frac{dz_1}{dt} = P_{12}z_2, \quad \frac{dz_2}{dt} = - P_{12}z_1, \quad \frac{dz_3}{dt} = 0, \quad \cdots\cdots \frac{dz_n}{dt} = 0,
$$

which may be integrated by one quadrature.

It appears from the preceding investigations that the problem of reduction of Λ_n to Λ_{n-m} amounts to this: *Given $m \cdot n$ cosines as functions of t, to calculate the remaining $n^2 - nm$ in terms of the given functions.* This is a purely algebraic problem, involving only the combination of $m - 1$ known transformations $S_n, S_{n-1}, \ldots. S_{n-m+1}$ where S_n is of the form (5). There exist $\infty^{\frac{n-2, n-}{2}}$ transformations which will transform Λ_n into Λ_{n-1} of which the transformation (5) is the simplest, corresponding, as it does, to the case where $\frac{n-2, n-1}{2}$ of the parameters $S_1 \ldots. S_{\frac{n.n-1}{2}}$ have been put equal to zero. The transformation

$S = S_n, S_{n-1} \ldots . S_{n-m+1}$ contains $(m-1)n - \frac{m(1+m)}{2}$ determinate parameters which are given functions of t; it follows therefore that in general there exist $\frac{n.n-1}{2} - \left\{n.m - 1 - \frac{m.(1+m)}{2}\right\} = \frac{(n-m)(n-m-1)}{2}$ arbitrary parameters, so that *there are* $\infty^{\frac{(n-m)(n-m-1)}{2}}$ *transformations which will reduce* Λ_n *to* Λ_{n-m}, m *sets of particular solutions being known.* In particular, if $m = n-1$ we find, since now $(n-m)(n-m-1) = 0$, that the transformation is uniquely determined by the $n-1$ given sets of solutions, which is indeed obvious, since $n-1$ sets of direction-cosines determine the remaining set of n cosines. In the transformation S all the $\frac{(n-m)(n-m-1)}{2}$ arbitrary parameters have been put equal to zero which insures that S is the simplest transformation possible.

If we compare the determinant of the system (14), (part II) with the determinant of the systems (9) and (11), we observe that the first is the product of the other two. In general, if we write the successive determinants of the transformations $S_n, S_{n-1} \ldots . S_{n-m+1}$ as follows:

$$D_n = \begin{vmatrix} \alpha_{11}^{(n)}\alpha_{12}^{(n)} & \ldots . & \alpha_{1n}^{(n)} \\ \alpha_{21}^{(n)}\alpha_{22}^{(n)} & \ldots . & \alpha_{2n} \\ \vdots & & \vdots \\ \alpha_{n1}^{(n)}\alpha_{n2}^{(n)} & \ldots . & \alpha_{nn}^{(n)} \end{vmatrix}, \quad D_{n-1} = \begin{vmatrix} \alpha_{11}^{(n-1)}\alpha_{12}^{(n-1)} & \ldots . & \alpha_{1n-1}^{(n-1)} & 0 \\ \alpha_{21}^{(n-1)}\alpha_{22}^{(n-1)} & \ldots . & \alpha_{2n-1}^{(n-1)} & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \alpha_{n-11}^{(n-1)}\alpha_{n-12}^{(n-1)} & \ldots . & \alpha_{n-1n-1}^{(n-1)} & 0 \\ 0 \quad 0 & \ldots . & 0 & 1 \end{vmatrix},$$

$$\ldots\ldots\ldots\ldots D_{n-m+1} = \begin{vmatrix} \alpha_{11}^{(n-m+1)}\alpha_{12}^{(n-m+1)} & \ldots . & \alpha_{1n-m+1}^{(n-m+1)} & 0 \ldots . 0 \\ \alpha_{21}^{(n-m+1)}\alpha_{22}^{(n-m+1)} & \ldots . & \alpha_{2n-m+1}^{(n-m+1)} & 0 \ldots . 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_{n-m+11}^{(n-m+1)}\alpha_{n-m+12}^{(n-m+1)} & \ldots . & \alpha_{n-m+1n-m+1}^{(n-m+1)} & 0 \ldots . 0 \\ 0 \quad 0 & \ldots . & 0 & 1 \ldots . 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 \quad 0 & \ldots . & 0 \ldots\ldots\ldots & 1 \end{vmatrix},$$

and from the product $D = D_n D_{n-1} \ldots . D_{n-m+1}$ we obtain the determinant of

the direction-cosines in the transformation S which transforms Λ_n into Λ_{n-m}. Let now m sets of direction-cosines be known particular solutions and let these be written

$$x^0_{n1},\ x^0_{n2} \cdots\cdots x^0_{nn};\ x^0_{n-11},\ x^0_{n-12},\ \cdots\cdots,\ x^0_{n-1n},\ \cdots\cdots,$$
$$\cdots\cdots x^0_{n-m+11},\ x^0_{n-m+12} \cdots\cdots x^0_{n-m+1n}.$$

In the transformation S_n which transforms Λ_n into Λ_{n-1} the last row of direction-cosines are identical with the first set of given solutions (see (5)), and all the remaining cosines in D_n are expressible in terms of these solutions. In D_{n-1} the $n-1^{st}$ row are particular solutions of the system Λ_{n-1} and the remaining cosines are expressible in terms of these; but by means of the transformation S_n^{-1} we may express $\alpha^{(n-1)}_{n-11}\ \alpha^{(n-1)}_{n-12} \cdots \alpha^{(n-1)}_{n-1n-1}, 0$ in terms of the second set of solutions $x^0_{n-11}\ x^0_{n-12} \cdots x^0_{n-1n}$, so that all the elements of D_{n-1} will be expressed in terms of these functions. Continuing the reasoning in this way, we find that the elements of D_{n-m+1} may all be expressed in terms of the last set of given solutions, viz.: $x^0_{n-m+11},\ x^0_{n-m+12} \cdots x^0_{n-m+1n}$. It follows that the product D will be an orthogonal determinant of the form

$$D = \begin{vmatrix} A_{11} & A_{12} \cdots\cdots & A_{1n} \\ A_{21} & A_{22} \cdots\cdots & A_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A_{n-m1} & A_{n-m} \cdots\cdots & A_{n-mn} \\ x^0_{n-m+11} & x^0_{n-m+12} \cdots & x^0_{n-m+1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x^0_{n1} & x^0_{n2} \cdots\cdots & x^0_{nn} \end{vmatrix}$$

where the A's are functions of the x's in the last m rows. If $m = n - 1$ the determinant reduces to the form

$$D = \begin{vmatrix} A_{11} & A_{12} \cdots & A_{1n} \\ x^0_{21} & x^0_{22} \cdots & x^0_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x^0_{n1} & x^0_{n2} \cdots & x^0_{nn} \end{vmatrix}$$

where $A_{11} \cdots A_{1n}$ are the minors of $x^0_{11},\ x^0_{12} \cdots x^0_{1n}$ in the determinant $|x^0_{11},\ x^2_{22},\ x^0_{33} \cdots x^0_{nn}|$.

If $m = n - 2$, we have the determinant

$$D = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ x^0_{31} & x^0_{32} & \dots & x^0_{3n} \\ \vdots & & & \\ x^0_{n1} & x^0_{n2} & \dots & x^0_{nn} \end{vmatrix},$$

and the general integral of (1) may be written down at once. We have

$$\begin{aligned} x_1 &= A_{11}x \sin \bar{\omega} + A_{21}x \cos \omega + C_1 x^0_{31} + C_2 x^0_{41} + \dots\dots\dots + C_{n-2}x^0_{n1}, \\ x_2 &= A_{12}x \sin \omega + A_{22}x \cos \omega + C_1 x^0_{32} + C_2 x^0_{42} + \dots\dots\dots + C_{n-2}x^0_{n2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= A_{1n}x \sin \omega + A_{2n}x \cos \omega + C_1 x^0_{3n} + C_2 x^0_{4n} + \dots\dots\dots + C_{n-2}x^0_{nn}, \end{aligned}$$

where $\omega = \int P_{12}dt$, and C_1, C_2 .. C_{n-2} are integration constants. A_{11}, A_{12} .. A_{1n} and A_{21}, A_{22} .... A_{2n} are all calculable in terms of the $n - 2$ given sets of particular solutions by a purely algebraic process as was explained above. For $m = n - 1$ we have the general integrals:

$$\begin{aligned} x_1 &= C_1 A_{11} + C^2 x^0_{21} + C_3 x^0_{31} + \dots\dots\dots C_n x^0_{n1}, \\ x_2 &= C_1 A_{12} + C_2 x^0_{22} + C_3 x^0_{32} + \dots\dots\dots C_n x^0_{n2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= C_1 A_{1n} + C_2 x^0_{2n} + \dots\dots\dots\dots\dots C_n x^0_{nn}, \end{aligned}$$

where A_{11}, A_{12} .... A_{1n} are the above-mentioned minors and C_1, C_2 .... C_n are the n integration constants.

The integration problem is thus reduced to its simplest form. It may be stated thus: *If m sets of particular solutions of the system Λ_n be known, the integration may be reduced to that of a system Λ_{n-m}.* If this last system be transformed by means of the transformation

$$\frac{y_1}{y_{n-m}} = \omega_1, \quad \frac{y_2}{y_{n-m}} = \omega_2, \quad \dots\dots\dots \quad \frac{y_{n-m-1}}{y_{n-m}} = \omega_{n-m-1},$$

into a generalized Riccati system in $n - m - 1$ variables, the above statement may be put in the form: *If m sets of particular solutions of Λ_n be known, the system may be reduced to a system of Riccati equations in $n - m - 1$ variables.*

For $m = n - 2$ we find the Riccati equation

$$\frac{d\omega}{dt} = P_{12}(1 + \omega^2),$$

which is integrable by one quadrature.

It should be noticed that the Riccati equation here obtained differs from the one obtained by Darboux's method in the particular case $n = 3$, $m = 1$; this equation is (Theorie des Surfaces, vol. I, p. 22)

$$\frac{d\sigma}{dt} = -ip_{12}\sigma + \frac{p_{13} - ip_{23}}{2} + \frac{p_{13} + ip_{23}}{2}\sigma^2,$$

and has no analogue for $n > 3$. It was the failure of this method when applied to the general system that led me to search for a method which would be applicable in general, and it has been shown that this method is chiefly based upon orthogonal, or more explicitly, Euclidean transformations.

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